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LETTER TO THE EDITOR

Application of the theory of Hill's equation to the study of the stability of periodic classical orbits

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Abstract. The theory of Hill's equation is applied to periodic orbits in a classical model of the magnetic hydrogen atom. It is shown how an infinite Hill determinant may be approximated and computed, thus giving the discriminant of the relevant Whittaker-Hill equation. This discriminant supplies information on the stability of the above periodic orbits which is much more easily obtained than by the numerical integrations of the orbits appearing in previous publications. The method is also applicable to other classical systems of current interest.

The stability of periodic orbits in the classical diamagnetic Kepler problem is important in the discussion of the transition from regular to chaotic motion, and of the relation between classical and quantum mechanical descriptions of the system. A number of studies have appeared recently which present the results of calculation of the stability of some of these orbits (Wintgen 1987, Al Laithy and Farmer 1987, Schweitzer *et al* 1988). These computations involved laborious numerical integrations of the orbits. However, results may be obtained in a much more elegant and economical fashion by means of the standard theory of Hill's equation. This method may also be applied to the study of periodic orbits in other Hamiltonian systems (e.g. the Hénon-Heiles system and x^2y^2 potential).

With a Larmor frame and the choice of $L_z = 0$ we have, in scaled units of position and momentum,

$$\tfrac{1}{2}(p_x^2 + p_z^2 + x^2 - 1/r) = E$$

$$\mathrm{d}p_x/\mathrm{d}t = -x - x/2r^3 \qquad \mathrm{d}p_z/\mathrm{d}t = -z/2r^3$$

where $p_x = \mathrm{d}x/\mathrm{d}t$ and $p_z = \mathrm{d}z/\mathrm{d}t$. The scaled energy E is identical with that used by the authors referred to above.

The singularity at the origin is removed by regularisation (see Szebehely 1967, Edmonds 1972) which introduces new coordinates u and v and a new 'time' τ:

$$x = u^2 - v^2 \qquad z = 2uv \qquad \mathrm{d}t/\mathrm{d}\tau = 4r.$$

The new 'energy' equation (where the 'energy' is identically equal to 2) is

$$\tfrac{1}{2}(p_u^2 + p_v^2) - 4(u^2 + v^2)E + 8(u^2 + v^2)u^2v^2 = 2$$

with the system of ODE

$$\mathrm{d}p_u/\mathrm{d}\tau - 8Eu + 16uv^2(2u^2 + v^2) = 0$$

$$\mathrm{d}p_v/\mathrm{d}\tau - 8Ev + 16u^2v(u^2 + 2v^2) = 0$$

$$p_u = \mathrm{d}u/\mathrm{d}\tau \qquad p_v = \mathrm{d}v/\mathrm{d}\tau.$$

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Consider motion near the z axis with $E<0$, using the regularised equations as above. (cf Pullen 1981.) We suppose $|v|_{\max}=\varepsilon$. Then

$$u''+8|E|u+O(\varepsilon^2)=0 \qquad v''+(8|E|+16u^4)v+O(\varepsilon^2)=0.$$

Take the solution of the first equation as $u=u_0\cos\Omega\tau$, where $\Omega^2=8|E|$. The energy equation gives $u_0=2/\Omega+O(\varepsilon)$.

Then

$$v''+[\Omega^2+96/\Omega^4+(128/\Omega^4)\cos 2\Omega\tau+(32/\Omega^4)\cos 4\Omega\tau]v=0$$

in this approximation.

This is the Whittaker-Hill equation (see Magnus and Winkler 1966) and was obtained by Edmonds (unpublished; see Pullen 1981); in the original work, use was made of a numerical tabulation of the regions of stability of the Whittaker-Hill equation by Klotter and Kotowski (1943). These data only covered a small range of the energy E, although the transition at $E=-0.39$ was predicted.

Many of the properties of the solutions of Hill's equation (the general case of a second-order linear differential equation with a periodic coefficient) are contained in the discriminant Δ, which is an analytic function of the Fourier coefficients of the periodic term. In our case the discriminant is given by

$$\Delta(\lambda)=2-4\sin^2(\pi\sqrt{\lambda}/2)C_0(\lambda)S_0(\lambda)$$

where $\nu=-16E^3$ $(E<0)$, $\lambda=1+3/\nu$ and C_0, S_0 are five-diagonal semi-infinite determinants with (apart from the first three rows) the elements of the nth row having non-zero values $1/2d_n$, $2/d_n$, 1, $2/d_n$, $1/2d_n$, where

$$d_n=3-(4n^2-1)\nu.$$

C_0 and S_0 may be obtained quickly by exponential (Richardson) extrapolation in $1/n_t$ of a few truncated subdeterminants of dimension n_t; the process is governed by the value of n for which d_n is nearest to 0. The size of determinant needed increases as E approaches 0. In the calculations reported here, the maximum dimension used was 810. The banded structure of the truncated determinants C_0 and S_0 permits the use of Wilkinson's efficient program BANDET1 (see Martin and Wilkinson 1967). The calculations were programmed in Lightspeed Pascal and executed on an Apple Macintosh computer.

Consideration of the Hill theory leads us to replace the variable E by

$$\mu=\tfrac{1}{2}[1-3/(16E^3)]^{1/2}$$

see figure 1.

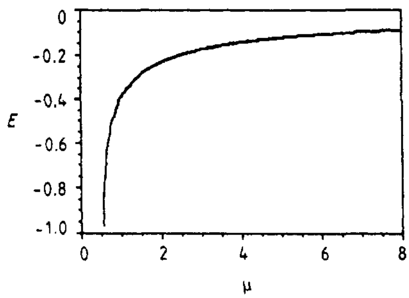

Figure 1. A plot of energy E against μ.

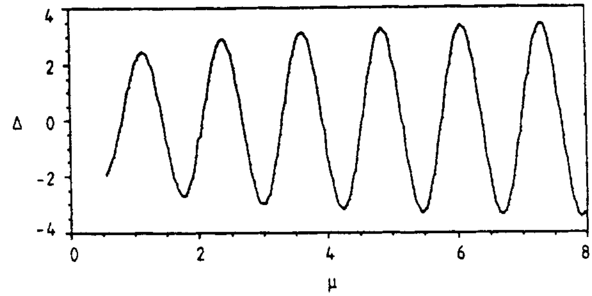

Figure 2. The discriminant Δ plotted as a function of μ for $E=-1, \ldots, -0.09$.

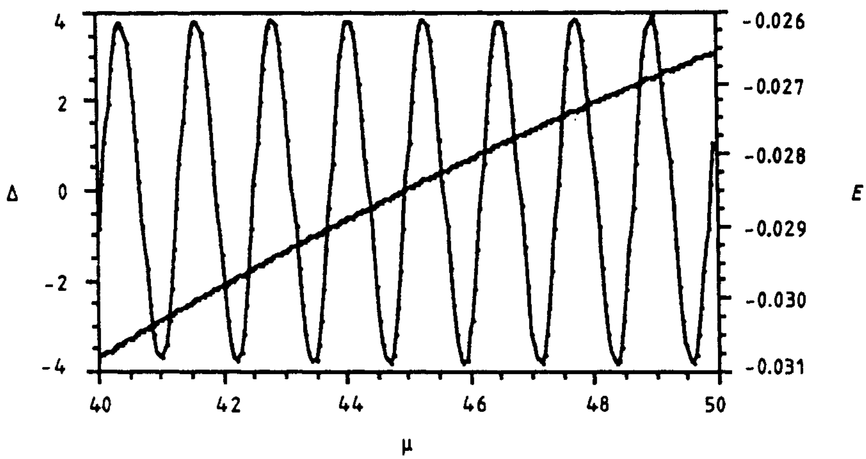

Figure 3. The discriminant Δ plotted as a function of μ for $E=-0.031, \ldots, -0.027$.

For all but small values of μ (corresponding to large negative E) Δ is very close to $a(\mu)\cos(b\mu+c)$ where $a(\mu)$ varies slowly with μ, and b and c are effectively constant (see figures 2 and 3). This result may be compared with other results in bifurcation theory (see Feigenbaum 1978, Bountis 1981) since the Feigenbaum ratio for μ as μ increases (and also for E more slowly as E tends to 0) goes rapidly to unity.

Regions of stability are given by $|\Delta|<2$, the regions of instability by $|\Delta|>2$. The Lyapunov exponent for unstable regions is $\beta=(1/\pi)\cosh^{-1}(\Delta/2)$ (see figure 4). Calculations of E values bounding stable regions agree with those obtained by Wintgen and others and have been extended close to $E=0$. The residue R computed by Al

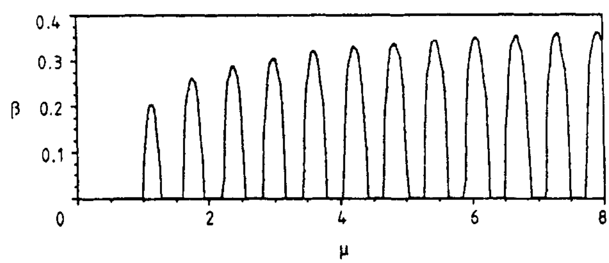

Figure 4. The Lyapunov exponent β plotted against μ for $E=-1.0, \ldots, -0.09$.

Laithy is essentially $2-\Delta^2$; it corresponds to the mapping T on the surface of section whereas Δ is related to the mapping $\sqrt{T}$ (cf the review by Greene 1979). The use of the theory of Hill's equation gives reliable results with less computing than other methods and may be extended to other periodic orbits of the H atom; however, we may not always have the advantage of banded determinants. For planar orbits an elliptic function complicates the problem and in any case we know already that there is only one region of stability and one of instability; the transition is at $E=-0.127$.

The above method of studying periodic orbits may be applied to other cases. The similar system with a non-linear term x^2y^2 in the potential studied by Pullen and Edmonds (1981) and Meyer (1986) gives rise to a Mathieu equation in place of a Whittaker-Hill equation. The well known Hénon-Heiles system is more complicated since elliptic functions and a non-linear form of Hill's equation appear.

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